Part 4: Functions

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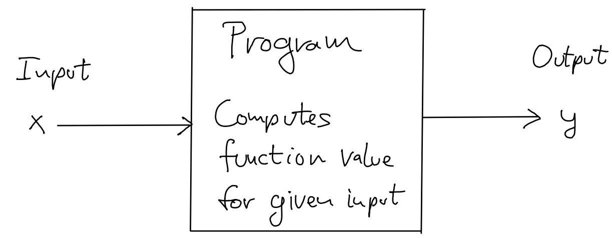
Functions

Lecture 20 Functions are a central concept in mathematics and computer science.

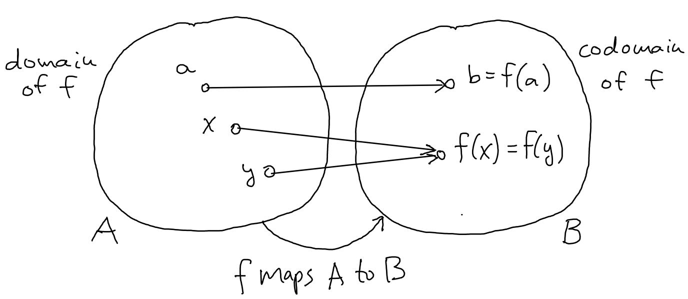
The general idea is to relate arguments (or inputs) with unique values (or outputs).

This succinctly characterizes what computers do:



Examples:

Input	Output
$x \in \mathbb{Q}$	$x \cdot x$
$x \in \mathbb{N}$	x is odd
$x \in \mathbb{N} \backslash \{0\}$	prime factorization of x
integer array A	sorted A
chess position x	move in position x
weather data d	weather forecast based on d



Definition: A function (or mapping) from A to B is a relation f from A to B such that

1. dom$(f) = A$, and
2. if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$

We write $f : A \to B$ which is read "f is a function from A to B", of "f maps A to B". B is called the **codomain of f**. In case where $A = B$, we say f is a **function on A**.

For $f : A \to B$ we write $y = f(x)$ when $(x, y) \in f$. We say y is the **value (or image) of f at x** and that x is the **pre-image of y under f**.

Important:

$f(x)$ does not denote a function!

f does. $f(x)$ denotes the image of x under f, which is an element of the codomain of f.

Sadly, in practice this distinction is often blurred.

E.g., when functions are described as $x+1$ or z^2, what is (maybe) meant is this:

$f : \mathbb{R} \to \mathbb{R}$, with $f(x) = x + 1$, and

$g : \mathbb{N} \to \mathbb{N}$, with $g(z) = z^2$.

Note, that we can't be sure about the domains!

Similarly, expressions like

$$f = x + 1$$

where f is meant to be a function and x is a number are mathematical nonsense.

Whenever you use $=$ in your derivations, make sure that both sides refer to objects of the same type.

Function vs. Relation Examples

$G = \{(x, y) \in \mathbb{R}^2 \mid y + x = 1\}$ with domain $\mathbb{R}$.

Is this a function?

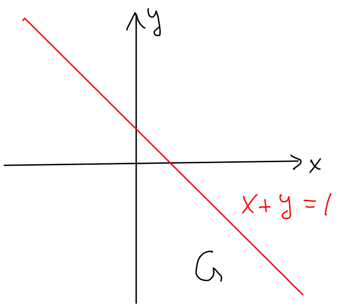
Yes, because for each x value there exists exactly one value y such that $y + x = 1$, namely $y = 1 - x$.

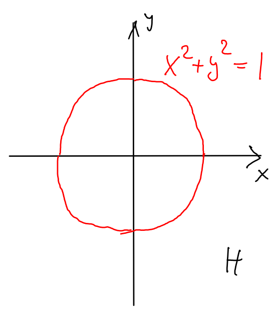
$H = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ with domain $[-1, 1]$.

Is this a function? No, because

$$(0, -1), (0, 1) \in H,$$

i.e. for $x = 0$ there exist two distinct values y_1, y_2 with $(x, y_i) \in H$, which means H is not a function.

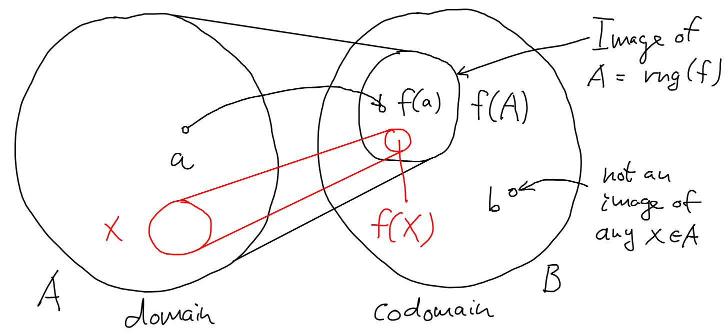




So, a function relates exactly one codomain element to each element of the domain.

Why do we differentiate between codomain and range?

Because we don't require all elements of the codomain to be images!



Definition: Let $f : A \to B$ and $S \subseteq A$. The **image** of S is defined as follows:

$$f(S) := \{b \in B \mid \exists a \in S : b = f(a)$$

Consequently, $f(S) \subseteq B$ and $\text{rng}(f) = f(A)$.

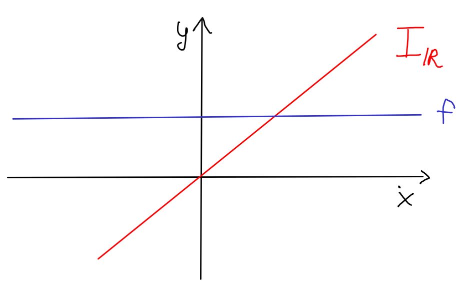
Function Examples

Let A be any set. The identity relation on A given by

$$I_A = \{(a, a) \mid a \in A\}$$

is in fact a function on A — the **identity function**. Why?

So, we can write $a = I_A(a)$, which holds for each $a \in A$.



Another important function type is **constant functions** $f : A \to B$, given by

$$f(a) = c$$

for each $a \in A$ and some fixed (constant) $c \in B$.

Example: $f(x) = 1$ for all $x \in \mathbb{R}$

Characteristic Functions

Let U be a set — called the "universe" — and $A \subseteq U$.

Define

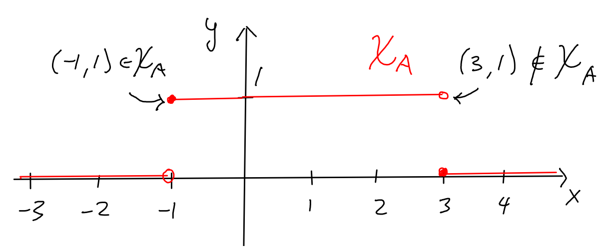
$$\chi_A : U \to [0, 1]$$

by

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

which is called the **characteristic function of set *A***. (χ: greek letter chi, lower case X, pronounced kai)

Example: $U = \mathbb{R}$ and $A = [-1, 3)$

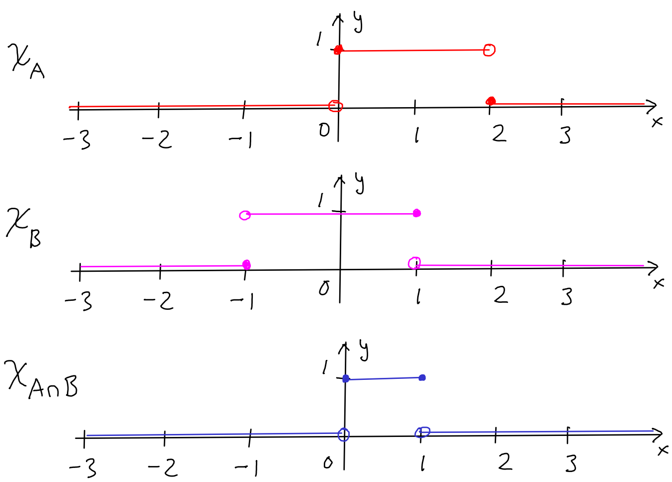


Theorem: Let $A, B \subseteq U$ and $x \in U$, then

1. $\chi_{A^c}(x) = 1 - \chi_A(x)$ (complement w.r.t. U)
2. $\chi_{A \cap B}(x) = \chi_A(x) \cdot \chi_B(x)$
3. $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_A(x) \cdot \chi_B(x)$

Proof: Exercise.

Illustration: $U = \mathbb{R}$, $A = [0, 2)$, $B = (-1, 1]$



Floor and Ceiling Functions

Two functions — floor and ceiling — that often occur in the analysis of algorithms map real numbers to integers like so:

$\lfloor x \rfloor :=$ biggest integer $\leq x$ ("floor" or "round down")

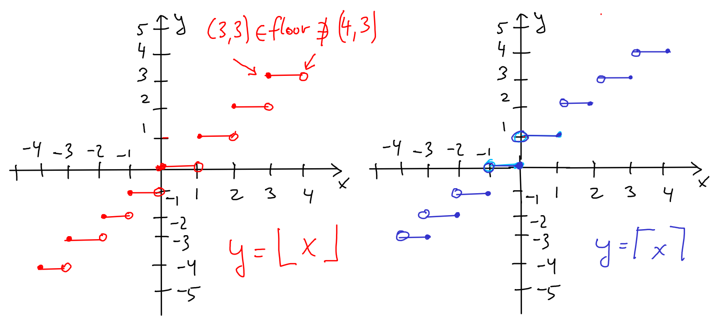
$\lceil x \rceil :=$ smallest integer $\geq x$ ("ceiling" or "round up")

Examples:

$\lfloor 3 \rfloor = \lceil 3 \rceil = 3$ $\quad \lfloor 3.4 \rfloor = 3$ $\quad \lceil 3.4 \rceil = 4$

$\lfloor -3.4 \rfloor = -4$ $\quad \lceil -3.4 \rceil = -3$

Graphs of floor and ceiling:



Theorem: Let $x \in \mathbb{R}$ and $k \in \mathbb{Z}$, then

1. $\lfloor x \rfloor = k$ iff $k \leq x < k+1$
2. $\lceil x \rceil = k$ iff $k - 1 < x \leq k$
3. $\lfloor x \rfloor \leq x \leq \lceil x \rceil$
4. $\lfloor x \rfloor > x - 1$ and $\lceil x \rceil < x + 1$
5. $\lfloor x + k \rfloor = \lfloor x \rfloor + k$ and $\lceil x + k \rceil = \lceil x \rceil + k$
6. $\lfloor \frac{k}{2} \rfloor + \lceil \frac{k}{2} \rceil = k$
7. $\lfloor x \rfloor = -\lceil -x \rceil$ and $\lceil x \rceil = -\lfloor -x \rfloor$

Proof:

5. Let $l = \lfloor x + k \rfloor$. Then with (1.), $l \leq x + k < l + 1$

(subtract k from all terms, inequalities stay valid)

$\Rightarrow (l - k) \leq x < (l - k) + 1$

$\overset{(1.)}{\Rightarrow} l - k = \lfloor x \rfloor$

$\Rightarrow l = \lfloor x \rfloor + k \quad \Rightarrow \lfloor x + k \rfloor = \lfloor x \rfloor + k$

The proof of the second part is analogous.

7. Let $l = \lceil -x \rceil$

$\overset{(2.)}{\Rightarrow} l - 1 < -x \leq l$ $\quad (\cdot(-1)$: flip inequalities)

$\Rightarrow -l + 1 > x \geq -l$ $\quad$ (reverse order)

$\Rightarrow -l \leq x < -l + 1$

$\overset{(1.)}{\Rightarrow} -l = \lfloor x \rfloor$

$\Rightarrow -\lceil -x \rceil = \lfloor x \rfloor$

Again, the second part can be shown analogously.

Other: Exercise.

Sequences

A function with domain $\mathbb{N}$ (or $\mathbb{N}_{>0} := \mathbb{N}\backslash\{0\}$) is called an **infinite sequence**.

Example: the infinite sequence of numbers

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4}, \ldots$$

can be described as function $a : \mathbb{N}_{>0} \to \mathbb{R}$ with

$$a(i) = \frac{1}{i}$$

Infinite sequences are commonly denoted as

$$(a_i)_{i=1}^{\infty}$$

where $a_i := a(i)$. Calculus studies properties of such sequences, like boundedness or convergence.

Finite sequences can be described as functions with domain $\{1 \ldots k\}$ for some $k \in \mathbb{N}_{>0}$.

E.g. $(b_i)_{i=1}^{5} = (2, 4, 6, 8, 10)$, with $b_i := b(i) = 2 \cdot i$

Functions with Multiple Arguments

Lecture 21

Up to now we have only considered functions with one argument, but often it is convenient to allow two or more.

Consider the expression: $3 + 2$

It denotes the application of function $+$ to two integer arguments: 2 and 3.

Using our notation that names the function first and can only deal with one argument we could write it as

$$+(\ (2, 3)\ )$$

or

$$+(x),\ \text{with } x = (2, 3)$$

where $+$ is the symbol for the addition function that as argument receives a pair of integers, and returns their sum as a result, i.e.

$$+ : \mathbb{Z}^2 \to \mathbb{Z},\ \text{with } +(\ (a, b)\ ) = \text{sum of } a \text{ and } b$$

Thus, we only have to deal with one-argument functions, even if inputs consist of multiple values.

However, writing double pairs of parenthesis when applying a function to a tuple is cumbersome and doesn't add any value apart from mathematical rigor.

In practice, the second parenthesis pair is therefore usually dropped with the understanding that n arguments separated by commas correspond to one argument that is an n-tupel.

E.g.

$$f(x_1, x_2, x_3) \text{ denotes } f(\ (x_1, x_2, x_3)\ )$$

Definition:
Assuming the flattened argument representation we just described, functions with more than one arguments are called **multivariate functions** and single-argument functions are called **univariate functions**.

Prefix, Infix, Postfix Notation

If we have one or more than two arguments we have the choice of either using **prefix** or **postfix** notation for functions:

$$f(x_1, x_2, \ldots, x_n) \text{ or } (x_1, x_2, \ldots, x_3)f$$

i.e. we name the function symbol first or last, repectively.

Postfix notation looks odd, but is in fact used in Hewlett Packard calculators, where you type

5 ENTER 3 ENTER 2 + +

to compute $5 + (3 + 2)$, and so called stack-machines (e.g. the Java Virtual Machine), in which all arguments are pushed on a last-in-first-out data structure (called stack), and operators (another name for functions) pop arguments from the stack and push the result back onto the stack.

With two arguments, we have a convenient third option: **infix** notation, which names functions in between their arguments similar to how we used it for relations $(aR\,b)$:

$$x + y \quad x - y \quad x \cdot y \quad x/y$$

Surjective Functions

Definition: A function $f : A \to B$ is **surjective** (or **onto B**) iff $\mathrm{Rng}(f) = B$. If f is surjective we write $f : A \overset{\text{surj}}{\to} B$.

Equivalently, surjective functions can be defined as functions for which there is **at least one pre-image** for each element of the codomain.

Examples:

$f : \mathbb{R} \overset{\text{surj}}{\to} \mathbb{R}$, with $f(x) = x$, is surjective because for each $y \in \mathbb{R}$ there is an $x \in \mathbb{R}$ — namely $x = y$ with $y = f(x)$.

$g : \mathbb{R} \to \mathbb{R}$, with $g(x) = x^2$, is not surjective because, $y = -1$ is not an image of any $x \in \mathbb{R}$, because all images are ≥ 0.

However, $h : \mathbb{R} \overset{\text{surj}}{\to} \mathbb{R}_{\geq 0}$, with $h(x) = x^2$ is surjective, because for $y \geq 0$, $x = \sqrt{y}$ is a pre-image.

Injective Functions

Definition: A function $f : A \to B$ is **injective** (or **one-to-one**) iff whenever $f(x) = f(y)$, then $x = y$. If f is injective we write $f : A \overset{\text{inj}}{\to} B$.

Alternatively, injective functions can be defined as functions for which there is **at most one pre-image** for each element of the codomain.

Examples:

$f : \mathbb{R} \overset{\text{inj}}{\to} \mathbb{R}$, with $f(x) = x$, because for each $y \in \mathbb{R}$ there is exactly one $x \in \mathbb{R}$ — namely $x = y$ — with $y = f(x)$.

$g : \mathbb{R} \to \mathbb{R}$, with $g(x) = x^2$, is not injective because, $y = 1$ has two pre-images: $x = -1$ and $x = 1$

However, $h : \mathbb{R}_{\geq 0} \overset{\text{inj}}{\to} \mathbb{R}$, with $h(x) = x^2$ is injective, because $y \geq 0$ has exactly one pre-image ($x = \sqrt{y}$) and $y < 0$ has none.

Bijective Functions

Definition: A function $f : A \to B$ is a **bijective** (or **one-to-one correspondence**) iff f is surjective and injective. If f is bijective we write $f : A \leftrightarrow B$.

Alternatively, bijective functions can be defined as functions for which there is **exactly one pre-image** for each element of the codomain.

Examples:

$f : \mathbb{R} \leftrightarrow \mathbb{R}$, with $f(x) = x$, is bijective because it is injective and surjective, as we have seen before.

$g : \mathbb{R} \to \mathbb{R}$, with $g(x) = x^2$, is not bijective, because it is not surjective.

However, $h : \mathbb{R}_{\geq 0} \leftrightarrow \mathbb{R}_{\geq 0}$, with $h(x) = x^2$ is bijective, because it is both injective and surjective.

New Functions from Old

Because functions are specialized relations, the earlier definitions of the inverse and composition of relations apply.

However, the nature of functions will allow us to simplify definitions and prove theorems that don't hold for relations in general.

Function Composition

Definition: For functions $g : A \to B$ and $f : B \to C$ the **composite of f and g** is the relation from A to C:

$$f \circ g = \{(x, z) \mid \exists y \in B : (x, y) \in g \wedge (y, z) \in f\}$$

The composite relation is always a function itself, as we will soon prove.

We can take advantage of the fact that each element of the domain of a function has a unique image to simplify the notation for composition.

Let $g : A \to B$ and $f : B \to C$.

Because

$$(x, y) \in g \text{ and } (y, z) \in f$$

can be written in the form

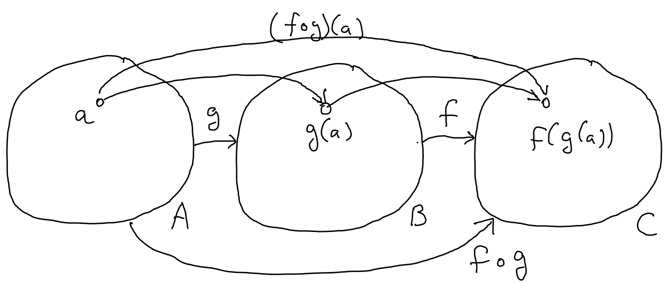
$$y = g(x) \text{ and } z = f(y),$$

we can write $z = f(g(x))$, i.e.

$$(f \circ g)(x) = f(g(x))$$

With this observation, we have an alternative definition of function composition:

$$f \circ g = \{(x, f(g(x))) \mid x \in A\}$$



Theorem: Let A, B and C be sets and $g : A \to B$ and $f : B \to C$. Then

1. $f \circ g$ is a function from A to C, and
2. Dom$(f \circ g) = A$.

Proof: 1. From Part 3 we know that $f \circ g$ is a relation from A to C.

To show that $f \circ g$ is a function from A to C, let $(x, z) \in f \circ g$ and $(x, z') \in f \circ g$.

We must prove that $z = z'$.

Because

$$(x, y), (x, y') \in f \circ g$$

there exist $b, b' \in B$ with

$$(x, b) \in g, (b, y) \in f \text{ and } (x, b') \in g, (b', y') \in f.$$

Since g is a function, $b = b'$.

Therefore, $(b, y) \in f$ and $(b, y') \in f$, which implies $y = y'$, because f is a function.

2. Exercise $\square$

Theorem: Let A, B, C, and D be sets and $h : A \to B, g : B \to C$, and $f : C \to D$. Then

$$(f \circ g) \circ h = f \circ (g \circ h)$$

and their domain is A.

Proof: The result follows from the associativity of composing relations and the previous Theorem which implies that the domain of both sides is A. $\square$

Example 1: Consider function f on $\mathbb{R}$ given by

$$f(x) = (x + 1)^2$$

Then

$$f = h \circ g$$

where g, h are functions on $\mathbb{R}$ with

$$g(x) = x + 1$$
$$h(x) = x^2$$

Example 2: Consider function f on $\mathbb{R}$ given by

$$f(x) = x + x^2$$

Then

$$f = t \circ h \circ g$$

where $g : \mathbb{R} \to \mathbb{R}^2,\ h : \mathbb{R}^2 \to \mathbb{R}^2,\ t : \mathbb{R}^2 \to \mathbb{R}$ with

$$g(x) = (x, x)$$
$$h(x, y) = (x, y^2)$$
$$t(x, y) = x + y$$

Example 3: Consider functions g, h on $\mathbb{R}$ given by

$$g(x) = 3x$$
$$h(x) = x/3$$

Then

$$f = h \circ g = I_{\mathbb{R}}$$

because $f(x) = h(g(x)) = (3x)/3 = x$.

Example 4: Consider functions f, g, h on $\mathbb{R}$ given by

$$f(x) = x + 1$$
$$g(x) = 2x$$
$$h(x) = x^2$$

Then

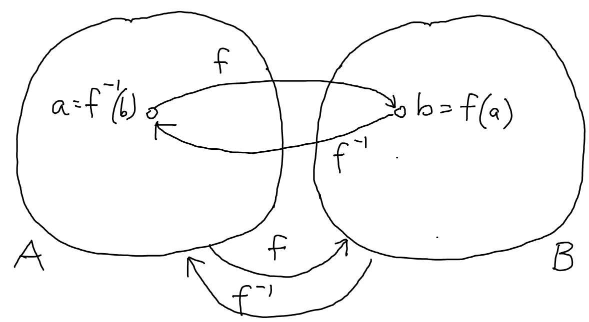
$$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = (f \circ g)(x^2)$$
$$= f(g(x^2)) = f(2x^2) = 2x^2 + 1$$

The Inverse of a Function

Lecture 22

Definition: For a function $f : A \to B$, the **inverse of f** is the relation from B to A:

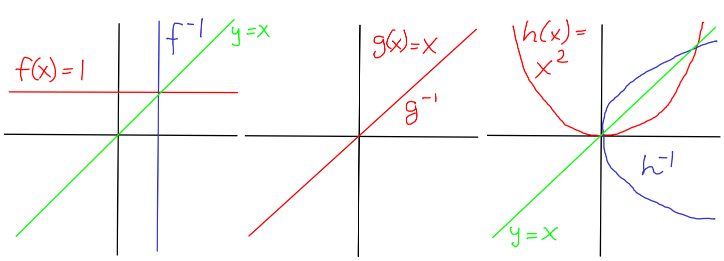
$$f^{-1} = \{(y, x) \mid (x, y) \in f\}$$



In general, f^{-1} is a relation, and might not be a function.

Examples

$f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 1$, $\quad f^{-1} = \{(1, x) \mid x \in \mathbb{R}\}$
$g : \mathbb{R} \to \mathbb{R}$ with $g(x) = x$, $\quad g^{-1} = \{(x, x) \mid x \in \mathbb{R}\}$
$h : \mathbb{R} \to \mathbb{R}$ with $h(x) = x^2$, $\; h^{-1} = \{(x^2, x) \mid x \in \mathbb{R}\}$



The inverse of a function on $\mathbb{R}$ can be obtained by mirroring all points on the diagonal $y = x$.

g^{-1} is a function, but f^{-1} and h^{-1} are not, because $(1, 1), (1, -1) \in f^{-1}$ and h^{-1}.

Pointwise Function Definition

Consider functions $f, g : A \to \mathbb{R}$.

From f, g we can construct a new function h pointwise by defining for each $x \in A$:

$$h(x) = f(x) + g(x)$$

or

$$h(x) = f(x) \cdot g(x)$$

or based on any other function $t : \mathbb{R}^2 \to \mathbb{R}$:

$$h(x) = t(f(x), g(x))$$

If we construct h in this way, we write $h = f\ t\ g$, e.g. $h = f + g$ or $h = f \cdot g$

Note that $+, \cdot$ in this case denote operations on functions which refer to the functions with the same name from $\mathbb{R}^2$ to $\mathbb{R}$ that are used to construct them.

As an example, consider pointwise addition or multiplication of infinite sequences $a, b : \mathbb{N} \to \mathbb{R}$:

$c = a + b$, i.e. for all $i \in \mathbb{N}$:

$$c_i = a_i + b_i$$

$c = a \cdot b$, i.e. for all $i \in \mathbb{N}$:

$$c_i = a_i \cdot b_i$$

Examples:

$$\begin{array}{ll} a: & (0,1,2,0,1,2,0,1,2,\ldots) \\ b: & (1,1,1,1,1,1,1,1,1,\ldots) \\ c = a+b: & (1,2,3,1,2,3,1,2,3,\ldots) \end{array}$$

$$\begin{array}{ll} a: & (0,1,2,0,1,2,0,1,2,\ldots) \\ b: & (2,1,2,1,2,1,2,1,2,\ldots) \\ c = a \cdot b: & (0,1,4,0,2,2,0,1,4,\ldots) \end{array}$$

Characteristic functions are also defined pointwise.

Some Basic Theorems About Functions

Theorem: Let $g : A \to B$ and $f : B \to C$. Then

1. If f, g are surjective, so is $f \circ g$.
2. If f, g are injective, so is $f \circ g$.
3. If f, g are bijective, so is $f \circ g$.
4. If $f \circ g$ is surjective, so is f.
5. If $f \circ g$ is injective, so is g.
6. g^{-1} is a function from Rng(g) to A iff g is injective.
7. If g^{-1} is a function, then g^{-1} is injective.
8. If g is bijective, so is g^{-1}

Proof:

1.+5.+7.+8. Exercise

2. Suppose $f(g(x)) = f(g(y))$. Then $g(x) = g(y)$ because f is injective, and $x = y$ because g is injective.

3. Follows from 1. and 2.

4. $f \circ g$ surjective $\Rightarrow$ Rng$(f \circ g) = C$

$\Rightarrow \forall c \in C\ \exists a \in A : f(g(a)) = c$

$\Rightarrow \forall c \in C\ \exists b \in B : f(b) = c$ (choose $b = g(a)$)

$\Rightarrow$ Rng$(f) = C$

6. a) Prove g^{-1} is a function from Rng(g) to $A \Rightarrow g$ is injective.

Suppose $g(x) = g(y) = z$

$\Rightarrow (z, x), (z, y) \in g^{-1}$

$\Rightarrow x = y$ (because g^{-1} is a function)

b) Prove: g is injective $\Rightarrow$ g^{-1} is a function from Rng(g) to A

Dom(g^{-1}) = Rng(g) and Rng(g^{-1}) = Dom$(g) = A$, therefore g^{-1} is a relation from Rng(g) to A.

Let $(x, y), (x, z) \in g^{-1}$. Then $(y, x), (z, x) \in g$ and $y = z$ because g is injective. This means g^{-1} is a function. $\square$